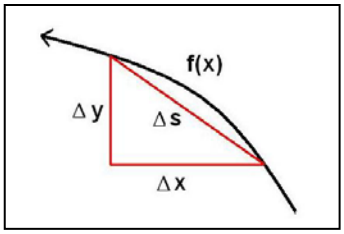
Calculating Arc Lengths of Simple Functions- I



Spirals are found in many different places in astronomy, from the shape of the arms in a 'spiral' galaxy, to the trajectory of a spacecraft traveling outward from Earth's orbit at constant velocity. Figuring out spiral lengths requires a bit of calculus. Here's how it's done:

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Step 1: Study the figure above, and use the Pythagorean Theorem to determine the hypotenuse length in terms of the other two sides. It should look like the equation to the left.

$$\Delta s = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

Step 2: Factor out the **Δx** to get a new formula.

$$S = \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Step 3: Following the basic techniques of calculus, 'take the limit' and allow the deltas to become differentials, then use the integral calculus to sum-up all of the differentials along the curve defined by y = F(x), and between points A and B, to get the fundamental arc-length formula.

$$S = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

The arc length formula can be re-written in polar coordinates too. In this case, the function, y = F(x) has been replaced by the polar function **r(θ).**

Problem 1) Find the arclength for the line y = mx + b from x=3 to x=10

Problem 2) Find the arclength for the parabolic arc defined by $y = x^2$ from x=1 to x=5.

Problem 3) Find the arclength for the logarithmic spiral $R(\theta) = e^{\mathbf{b\theta}}$ from $\theta = 0$ to $\theta = 4\pi$ if b = 1/2.

Problem 4) The spiral track on a CDROM is defined by the simple formula R = kq, where k represents the width of each track of data. If k = 1.5 microns, how long is the spiral track, in meters, for a standard 6-cm disk if the hub space is also used?

Answer Key

Problem 1) $dy/dx = m$, so the integrand becomes $(1 + m^2)^{1/2}$ dx. Because m is a constant independent of x, the integral is just $(1 + m^2)^{1/2}$ (10 - 3) = $7 (1+ m^2)^{1/2}$.

Problem 2) $dy/dx = 2x$ and the integrand becomes $(1 + 4x^2)^{1/2}$ dx. This can be integrated by using the substitution $2x = \sinh(u)$, and $dx = (1/2)\cosh(u)\, du$, so that the integrand becomes $1/2 \cosh^2(u)\, du$. This is a fundamental integral with the solution **1/2 [sinh(2u)/4 + u/2 + C]**.

Limits: The limits go from x=1 to x=5, but since $x = 1/2 \sinh(u)$, the limits re-expressed in terms of u become $u_1 = \sinh^{-1}(2) = 1.44$ and $u_2 = \sinh^{-1}(10) = 3.00$ so evaluating the definite integral leads to 1/2 (sinh(6.00)/4 + 3.00/2) - 1/2 (sinh(2.88)/4 + 1.44/2) = 25.96 - 1.47 = 24.49.

Problem 3) We use the polar form of the arclength formula. First perform the differentiation of $r(\theta)$ to get $dr/d\theta = b\, e^{b\theta}$. Then after substitution, the integrand becomes $(e^{2b\theta} + b^2 e^{2b\theta})^{1/2}\, d\theta$ which after simplification then becomes $e^{b\theta} (1 + b^2)^{1/2}\, d\theta$. This is easily integrated to get $(1/b)\,(1 + b^2)^{1/2} e^{b\theta}$ + **C**. Since b = 1/2, we get the simpler form $2.24\, e^{\theta/2} + C$. This can be evaluated between the two limits for θ to get 2.24 (534.86 - 1) = **1,195.85.**

Problem 4) Because $r = k\theta$, the integrand becomes $(k^2\theta^2 + k^2)^{1/2}\, d\theta$ or $k (1 + \theta^2)^{1/2}\, d\theta$. This can be simplified using the hyperbolic trig identity $1 + \sinh^2(x) = \cosh^2(x)$ where we have used the substitution $\theta = \sinh(x)$. This also means that $d\theta = \cosh(x)\, dx$. Then the integrand becomes $\cosh^2(x)\, dx$. The integral is then a fundamental integral with the solution **k [sinh(2x)/4 + x/2 + C]**.

Limits: How many radians does the spiral take up? 2π x 6 cm/1.6 microns = 2π x 60000/1.5 = 80,000 π. This means that the integral will have limits from 0 to 80,000 π. But q = sinh(x) so the limits become x1 = 0 to x2 = $\sinh^{-1}(80{,}000\, \pi) = 13.13$. The definite integral is then 1.5 microns x [(sinh(26.26) +13.13/2 + C) - (sinh(0)/4 + 0/2 +C) = 0.0000015 meters (1.26×10^{11} + 6.56) = **189,000 meters or 189 kilometers!**